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# STRESS ANALYSIS OF A SEMI-INFINITE PLATE CONTAINING A REINFORCED NOTCH UNDER UNIFORM TENSION

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**Abstract**—A solution is presented for stresses in a semi-infinite plate containing a reinforced notch on one side under uniform tension. The relationships between the stress concentration factor and the dimensions of a reinforcing member are obtained. Comparison is made between theoretical and experimental results using a photoelastic method, and the appropriate dimensions of the reinforcing member are determined within a range where both of the results are in good agreement.

## NOTATION

| | |
|---|---|
| $z = x+iy$ | co-ordinates in the physical plane |
| $\zeta = \rho\, e^{i\theta}$ | co-ordinates in the mapped annulus plane |
| $\sigma$ | $= e^{i\theta}$ |
| $\theta$ | argument of $\zeta$ |
| $z = w(\zeta)$ | conformal transformation function |
| $\sigma_\rho, \sigma_\theta, \tau$ | stress components in the mapped annulus plane |
| $T$ | tension in the reinforcement |
| $f$ | applied stress at infinity in the $y$-direction |
| $A, t$ | cross-sectional area of reinforcement and thickness of the plate, respectively |
| $\nu$ | Poisson's ratio |
| $R, p, q$ | defined in equation (1) |
| $a, 2b$ | depth and breadth of a notch |
| $a_n, b_n$ | constants in the Laurent series, see equation (3) |
| $K$ | $= A/Rt$ |
| $\varphi(\zeta), \psi(\zeta)$ | complex stress potentials, see equation (3) |
| $P$ | $= 1+p^2+9q^2+(-2p+6pq)\cos 2\theta-6q\cos 4\theta$ |
| $H$ | $= 1-p^2-27q^2-12pq\cos 2\theta+6q\cos 4\theta$ |

## ANALYSIS

CONSIDER a semi-infinite plate with a single reinforced notch subjected to a uniform tension in the $y$-direction as illustrated in Fig. 1. Now, we introduce the mapping relation

$$z = w(\zeta) = R\left(\zeta+\frac{p}{\zeta}+\frac{q}{\zeta^3}\right) \tag{1}$$

where $R$, $p$ and $q$ are real constants and $R > 0$, $1 > |p| \geqq 0$ and $1 > |q| \geqq 0$.

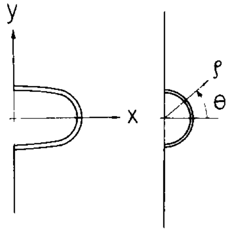


FIG. 1. Notched plate under tension.

The symbols used are the same as those of Muskhelishvili [1], who shows that the three stress-components may be given by

$$\left.\begin{aligned} \sigma_\rho+\sigma_\theta &= 2\left[\frac{\varphi'(\zeta)}{w'(\zeta)}+\frac{\bar\varphi'(\bar\zeta)}{\bar w'(\bar\zeta)}\right] \\ \sigma_\theta-\sigma_\rho+2i\tau &= 2\left[\bar w(\bar\zeta)\left\{\frac{\varphi'(\zeta)}{w'(\zeta)}\right\}'+\psi'(\zeta)\right]\zeta^2/\rho^2\bar w'(\bar\zeta) \end{aligned}\right\} \tag{2}$$

where the prime denotes differentiation with respect to the variable indicated.

The plate is subjected to a uniform tension in the direction of axis with the straight-edge free from loading. In order to satisfy these boundary conditions, we may write $\varphi(\zeta)$ and $\psi(\zeta)$ as follows:

$$\left.\begin{aligned} \varphi(\zeta) &= \frac{fR}{4}\left(\zeta+\frac{p}{\zeta}+\frac{q}{\zeta^3}\right)-fR\sum_0^\infty\left[\frac{a_n}{2n+2}\zeta^{-2n-2}+\frac{b_n}{2n+1}\zeta^{-2n-1}\right] \\ \psi(\zeta) &= \frac{fR}{2}\left(\zeta+\frac{p}{\zeta}+\frac{q}{\zeta^3}\right)+fR\left[\frac{w(\zeta)}{w'(\zeta)}\sum_0^\infty(a_n\zeta^{-2n-3}+b_n\zeta^{-2n-2}) \right. \\ &\qquad \left. +\sum_0^\infty\left\{\frac{a_n}{2n+2}\zeta^{-2n-2}-\frac{b_n}{2n+1}\zeta^{-2n-1}\right\}\right] \end{aligned}\right\} \tag{3}$$

where $a_n$ and $b_n$ are real constants.

$\varphi(\zeta)$ and $\psi(\zeta)$ can be expressed by the unknown coefficients $a_n$ and $b_n$, which can be determined by the boundary conditions at the edge of the reinforced notch.

The boundary conditions of the reinforced notch were discussed in detail by Wittrick [2]. By using the equilibrium condition of forces, the tension $T$ in the reinforcement was derived as follows:

$$T/2t = [\bar w'(\sigma)\varphi(\sigma)+w(\sigma)\bar\varphi'(\sigma)+\bar w'(\sigma)\bar\psi(\sigma)]/R\sigma(\sqrt{P}). \tag{4}$$

From the equilibrium of an element of reinforcement, he also introduced $T$ as follows:

$$T/2A = \left[\frac{\varphi'(\sigma)}{w'(\sigma)}+\frac{\bar\varphi'(\sigma)}{\bar w'(\sigma)}\right]/[1+(1+\nu)KHP^{-\frac{3}{2}}]. \tag{5}$$

From formulae (4) and (5), the following relation must be satisfied along the edge of the reinforced notch:

$$\bar w'(\sigma)\varphi(\sigma)+w(\sigma)\bar\varphi'(\sigma)+\bar w'(\sigma)\bar\psi(\sigma) = \left[\frac{\varphi'(\sigma)}{w'(\sigma)}+\frac{\bar\varphi'(\sigma)}{\bar w'(\sigma)}\right]R^2\sigma K/\{P^{-\frac{1}{2}}+(1+\nu)KHP^{-2}\}. \tag{6}$$

Substituting formula (3) into (6), we obtain the two expressions by comparing each term in the real and imaginary parts. Therefore, the coefficients $a_n$ and $b_n$ can be determined so as to satisfy these expressions.

## NUMERICAL SOLUTION

$P$ and $H$ are functions of $\theta$. It is thus impossible to obtain the unknown coefficients from equation (6) simply by a process of comparing coefficients except in particular cases. In fact, only an approximate solution can be obtained, by using a finite number of coefficients $a_n$ and $b_n$ and satisfying equation (6) at a certain number of points in the range of $0 \leqq \theta \leqq \pi/2$.

The series for $T/Af$ converges sufficiently for practical purposes by taking fifteen terms in each $a_n$ and $b_n$. Equal increments of $\theta$ are specified.

Now, several cases for the values of $p$ and $q$ will be analysed as an example. Various kinds of opening can be obtained by changing the values of $p$ and $q$, which are tabulated in Table 1. Shapes corresponding to these values are illustrated in Fig. 2 where $a/b$ is assumed to be 7/4.

TABLE 1

| No. | I | II | III | IV |
|---|---|---|---|---|
| $p$ | 0·2911 | 0·2770 | 0·2642 | 0·2563 |
| $q$ | 0·0672 | 0·0155 | −0·0315 | −0·0602 |
| $r/a$ | 0·10 | 0·25 | 0·57 | 1·00 |
| $a/b$ | | 7/4 | | |

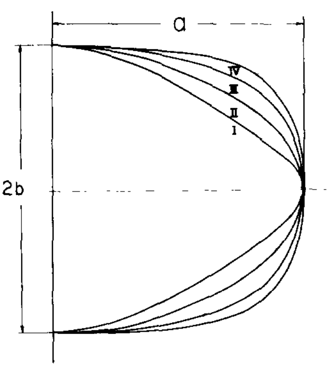


FIG. 2. U type notch. $a$ and $2b$ are depth and breadth of the notch, respectively.

With the values tabulated in Table 1, the relationships between $T/Af$ and $K$ are obtained. The small value of $K$ is chosen because of the fact that errors of stresses obtained will increase with a larger value of $K$ when the reinforcement is treated in one-dimensional configuration.

The relationship between $K$ and $T/Af$ at $y = 0$ with specified values of $r/a$ is illustrated in Fig. 3. $r$ is the radius of curvature of the reinforcement. The specified value of $K$ is chosen as 0, 0·1, 0·2 and 0·3, respectively. The stress concentration factors decrease considerably as the value of $K$ increases in the range of $0 \leqq K \leqq 0{\cdot}1$. The relationship between $r/a$ and $T/Af$ at $y = 0$ with specified values of $K$ is illustrated in Fig. 4. $T/Af$ decreases remarkably as the value of $r/a$ increases for $0 \leqq K \leqq 0{\cdot}1$ and decreases gradually for $r/a > 0{\cdot}3$ regardless of $K$.

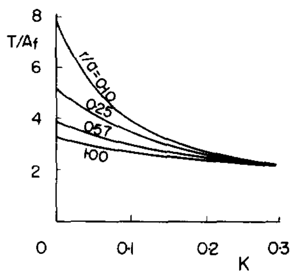


FIG. 3. Relationship between $K$ and $T/Af$ at $x = a$, $y = 0$.

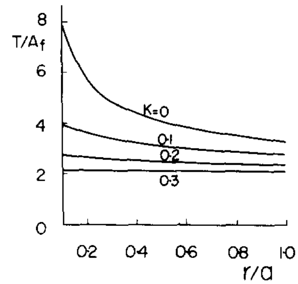


FIG. 4. Relationship between $r/a$ and $T/Af$ at $x = a$, $y =$

The distribution of $T/Af$ along the edge of the notch is illustrated in Fig. 5 when $r/a = 0{\cdot}57$ with the specified values of $K = 0, 0{\cdot}1, 0{\cdot}2$ and $0{\cdot}3$. It is seen that $T/Af$ decreases considerably as the distance from the bottom of the notch increases.

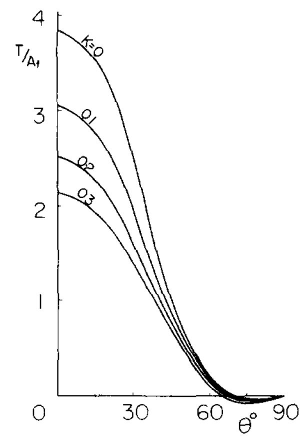


FIG. 5. Distribution of $T/Af$ around the edge of the notch for $r/a = 0{\cdot}57$.

## COMPARISON WITH THE RESULTS OF EXPERIMENTS

Stress measurements were carried out by the photoelastic method [3, 4] and the results are compared with the theoretical ones. Since it is difficult to make a specimen with a reinforced U-type notch, experiments were made on a specimen with a reinforced circular notch. The specimen is illustrated in Fig. 6. $h$ and $t$ are thickness of reinforcement and plate, respectively. The plate is made of epoxy resin and the reinforcement is of acrylic resin.

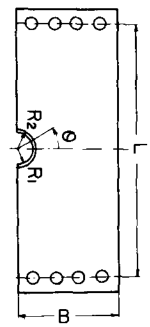


FIG. 6. Specimen.

The value of $B/R_2$ is assumed to be nine so as to prevent local disturbance of the state stress in the vicinity of the notch from the effect of the other free boundary. In the experiment, $L$, $B$, $R_1$, $R_2$ and $t$ were taken as 360, 160, 16, 17 and 6 mm, respectively. Reinforcements, of thickness 1 mm, were made of acrylic resin. $K$ $(= A/Rt)$ became 0·08 for $R = (R_1+R_2)/2$ and $h$ = 8 mm. Experiments were carried out for the case of thin reinforcement. As the reinforcement becomes thick, three-dimensional effects for the various stresses are introduced, which the author [5] previously analysed in detail.

Two reinforcements are joined to both surfaces of the plate and epoxy resin is used as a binding agent. The radius of the specimen is bored about 2 mm smaller beforehand. After curing, the specimen is machined to the required dimensions with a sharp tool. This procedure is followed so that the part of the specimen containing the initial strains (introduced around the notch during specimen manufacture) is cut off.

The experimental result of the distribution of $T/Af$ along the edge of a notch is indicated in Fig. 7 by the symbol "O". The theoretical value of $T/Af$, which is illustrated as a full line, was obtained using fifteen terms of $a_n$ and $b_n$, respectively. Both results are in good agreement. But the discrepancy between them becomes larger as the value of $K$ increases.

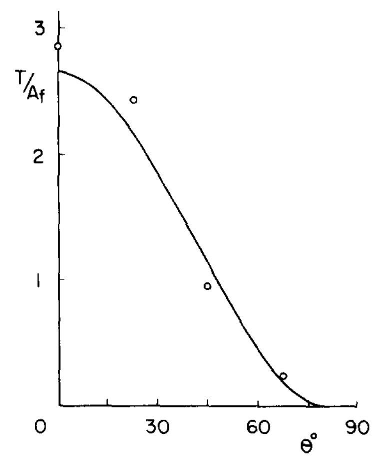


FIG. 7. Comparison between theoretical and experimental results for $K$ = 0·08.

## ON THE ACCURACY OF THE SOLUTION BY THE WITTRICK METHOD

Wittrick [2] treated a reinforcement in one-dimensional configuration. From his assumption, it will easily be understood that the accuracy of the solution by his method will decrease when the value of $K$ becomes large. Let us consider the range of $K$ where the theoretical and experimental results are in good agreement.

Comparison will be made between the theoretical results by Wittrick and Gurney [6] and the experimental ones by the author [4] when a plate with a reinforced circular hole is subjected to an axially symmetrical load. $\sigma_\theta/p$ at the inner edge of the reinforcing ring with specified value of $h/t = 2$ are illustrated in Fig. 8. In this case $K$ is calculated on the assumption of $R = (R_1+R_2)/2$. O indicates the experimental results by the author using a photoelastic method.

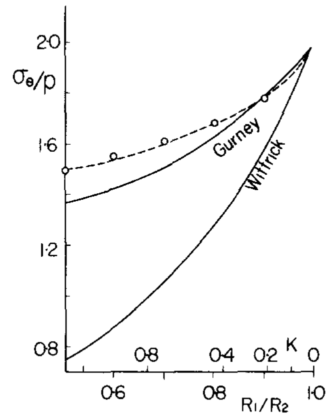


FIG. 8. Comparison between the theoretical results by Wittrick and Gurney and the experimental ones by the author for $h/t = 2$.

Although Gurney's assumption for the junction of the plate and the ring is only roughly approximate, it will easily be understood that his theoretical results will be good when $h/t$ is in the neighbourhood of 1. His results are in good agreement with the experimental ones by the author in the range $R_1/R_2 > 0{\cdot}8$. But as problems are treated two-dimensionally, it is very difficult to analyse them by his method, except for a circular hole.

In contrast, there is a large discrepancy between the theoretical results by Wittrick and the experimental ones. For $K = 1{\cdot}2$, the theoretical value is approximately half the experimental one. For $K = 0{\cdot}1$, there is about a 6 per cent discrepancy between them, which may be tolerated for practical purposes. The value of $K$ in the reinforced window of an aircraft approximates to 0·1. In such a case, his method will be useful for design calculations.

## CONCLUSIONS

The following conclusions can be drawn from the results of this work.

(1) The stress concentration factor $T/Af$ at $y = 0$ decreases remarkably as the value of $K$ increases for $0 \leqq K \leqq 0{\cdot}1$. This reduction becomes large when $r/a$ is small.

(2) $T/Af$ at $y = 0$ decreases considerably as the value of $r/a$ increases for $r/a < 0{\cdot}3$, but $T/Af$ is almost independent of $r/a$ when $K$ approaches the value 0·3.

(3) The distribution of $T/Af$ around the edge of the notch decreases considerably as the distance from the bottom of the notch increases.

(4) The accuracy of the solution by the Wittrick method becomes low as the value of $K$ increases. But the solution will be accurate enough for the practical purposes if $K < 0{\cdot}1$.

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**Résumé**—Une solution est présentée pour des contraintes dans une plaque semi-infinie contenant une entaille renforcée à un côté sous une tension uniforme. Les relations entre le facteur de concentration de contrainte et les dimensions d'un membre renforçant sont obtenues. Une comparaison est faite entre les résultats théoriques et expérimentaux employant une méthode photoélastique, et les dimensions appropriées du membre renforçant sont déterminées en une gamme où les deux résultats s'accordent.

**Zusammenfassung**—Eine Lösung wird gegeben für Spannungen in einer half-unendlichen Platte mit einer verstärkten Kerbe an einer Seite unter gleichförmiger Spannung. Die Verhältnisse von Spannungskonzentration und den Dimensionen der Verstärkungsteile werden gefunden. Theoretische und Versuchsresultate werden verglichen mittels einer photoelastischen Methode, und die Abmessungen des Verstärkerteiles werden bestimmt, für einen Bereich, wo die Resultate gute Übereinstimmung zeigen.

**Абстракт**—Предоставляется решение для напряжений в полубесконечной пластине, обладающей на одной стороне укреплённой выемкой пол однородным напряжением. Получены соотношения между фактором концентрации напряжения и размерами укрепляющей части. Сделано сравнение между теоретическим и экспериментальным результатами применением фотоэластического метода и определены соответствующие размеры укрепляющей части в области, где оба результата хорошо согласуются.